

# Preconditioning strategies for implicit MHD

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SOLUTION METHODS FOR LARGE-SCALE NONLINEAR PROBLEMS  
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# Outline

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# Introduction

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# Motivation for an implicit MHD solver

- The MHD formalism is a nonlinear system of **stiff equations**:
  - Elliptic stiffness (transport).
  - Hyperbolic stiffness (linear waves: magnetosonic, Alfvén, sound, whistler,...).
- Explicit methods: **straightforward but inefficient** (numerical stability).
- Semi-implicit methods: popular, efficient, but potentially inaccurate (linearization, splitting, simplifications in semi-implicit operator).
- Implicit methods: accurate and efficient, but of difficult implementation:
  - Non-linear couplings in equations.
  - Ill-conditioned matrices due to elliptic operators and stiff waves.
- Here, a viable, scalable preconditioner strategy for a Newton-Krylov implicit MHD solver is explored.

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## Newton-Krylov solver

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# Jacobian-Free Newton-Krylov Methods

- Objective: solve nonlinear system  $\vec{G}(\vec{x}^{n+1}) = \vec{0}$  efficiently.
- Converge nonlinear terms in  $\vec{G}(\vec{x}^{n+1}) = \vec{0}$  using Newton-Raphson method.
- Jacobian-free implementation:

$$\left( \frac{\partial \vec{G}}{\partial \vec{x}} \right)_k \vec{y} = \lim_{\epsilon \rightarrow 0} \frac{\vec{G}(\vec{x}_k + \epsilon \vec{y}) - \vec{G}(\vec{x}_k)}{\epsilon}$$

- Krylov method of choice: GMRES (nonsymmetric systems).
- Right preconditioning:

$$\vec{y} = P_k \vec{v} ; \vec{v} \rightarrow \text{Krylov vector}$$

$$J_k \vec{y} = \lim_{\epsilon \rightarrow 0} \frac{\vec{G}(\vec{x}_k + \epsilon \vec{y}) - \vec{G}(\vec{x}_k)}{\epsilon}$$

APPROXIMATIONS IN PRECONDITIONER DO NOT AFFECT ACCURACY OF CONVERGED SOLUTION; THEY ONLY AFFECT EFFICIENCY!

# Physics-based preconditioning

- It is an approximate solver strategy based on two fundamental elements:
  1. Approximations based on physical insight to obtain a diagonally-dominant formulation.
  2. The use of approximate (and possibly inaccurate) solver technologies (approx. MG, split, SI).
- Example: two coupled, first-order wave equations:

$$\partial_t u = \partial_x v, \quad \partial_t v = \partial_x u.$$

Differencing implicitly in time (with backward Euler for simplicity):

$$\begin{aligned} u^{n+1} &= u^n + \Delta t \partial_x v^{n+1}, \\ v^{n+1} &= v^n + \Delta t \partial_x u^{n+1}. \end{aligned}$$

Substitute the second equation into the first to obtain:

$$(I - \Delta t^2 \partial_{xx}) u^{n+1} = u^n + \Delta t \partial_x v^n.$$

- Connection with Schur complement:

$$\begin{bmatrix} I & -\Delta t \partial_x \\ -\Delta t \partial_x & I \end{bmatrix} = \begin{bmatrix} I & -\Delta t \partial_x \\ 0 & I \end{bmatrix} \begin{bmatrix} I - \Delta t^2 \partial_{xx} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -\Delta t \partial_x & I \end{bmatrix}.$$

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## 2D Resistive MHD

L. Chacón, D. A. Knoll and J. M. Finn, *J. Comput. Phys.*, 178, 15-36 (2002)

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## Equations

- The 2D reduced MHD equations (incompressible flow) are (Alfvénic units):

$$\nabla^2 \Phi = \omega \quad (1)$$

$$(\partial_t + \vec{v} \cdot \nabla - S^{-1} \nabla^2) \Psi + E_0 = 0 \quad (2)$$

$$(\partial_t + \vec{v} \cdot \nabla - Re^{-1} \nabla^2) \omega + S_\omega = \vec{B} \cdot \nabla (\nabla^2 \Psi) \quad (3)$$

where  $\vec{v} = \vec{z} \times \nabla \Phi$  ;  $\vec{B} = \vec{z} \times \nabla \Psi$ ,  $S = \eta^{-1}$  is the Lundquist number and  $Re$  the Reynolds number.

- This system supports the Alfvén wave, which is a fast normal mode of the system and limits the time step in explicit implementations.
- The domain is a rectangle of size  $L_x \times L_y$ .
- Differencing: second-order accurate in space and in time.

## Physics-based preconditioner for RMHD

- Presence of **stiff waves** (hyperbolic system) results in **ill-conditioned algebraic systems**:

$$J \sim \begin{pmatrix} k^2 & 0 & 1 \\ kv_A & \frac{1}{\Delta t} + kv_0 & 0 \\ k\omega'_0 & k^3 v_A & \frac{1}{\Delta t} + kv_0 \end{pmatrix}$$

- Physics-based preconditioner for **Alfvén wave** in RMHD model is constructed on the previous first-order wave equation template.
- However, **RMHD is a deceptively simple system**. Alfvén wave is propagated by:

$$\begin{aligned}\partial_t \delta\Psi &= \vec{B}_0 \cdot \nabla \delta\Phi \\ \partial_t \nabla^2 \delta\Phi &= \vec{B}_0 \cdot \nabla (\nabla^2 \delta\Psi)\end{aligned}$$

Presence of  $\partial_t \nabla^2$  complicates matters, because a direct substitution is not possible.

A WORKAROUND IS POSSIBLE!

## Physics-based preconditioner for RMHD (II)

- We linearize the RMHD system and eliminate  $\delta\omega$ :

$$L_{S_L} \delta\Psi = \theta \vec{B}_0 \cdot \nabla \delta\Phi - G_\Psi,$$

$$\left[ L_{Re} \nabla^2 - \theta (\vec{z} \times \nabla \omega_0) \cdot \nabla \right] \delta\Phi = \theta [(\vec{B}_0 \cdot \nabla) \nabla^2 - \vec{z} \times \nabla J_0 \cdot \nabla] \delta\Psi - G_\omega + L_{Re}(G_\Phi),$$

where  $L_\chi = \frac{1}{\Delta t} + \theta \vec{v}_0 \cdot \nabla - \theta \chi \nabla^2$  is the advection-diffusion operator.

- Crucial simplification (using  $\nabla \cdot \vec{B}_0 = 0$ ):

$$[(\vec{B}_0 \cdot \nabla) \nabla^2 - (\vec{z} \times \nabla J_0) \cdot \nabla] \delta\Psi = \nabla \nabla : [\vec{B}_0 \nabla \delta\Psi - \delta\Psi \nabla \vec{B}_0] \sim \nabla^2 (\vec{B}_0 \cdot \nabla) \delta\Psi.$$

Exact for uniform  $\vec{B}_0$ . Same is possible for  $\vec{v}_0$ . Laplacian operator is extracted.

- We rearrange system to find:

$$L_{S_L} \delta\Psi = \theta \vec{B}_0 \cdot \nabla \delta\Phi - G_\Psi,$$

$$L_{Re} \delta\Phi = \theta \vec{B}_0 \cdot \nabla \delta\Psi + \underbrace{\nabla^{-2} [-G_\omega - L_{Re}(-G_\Phi)]}_{rhs_\Phi}.$$

LAPLACIAN OPERATOR HAS BEEN REMOVED.

NOW WE CAN ATTEMPT TO USE THE TEMPLATE OF THE SIMPLE HYPERBOLIC SYSTEM.  
HOWEVER, THIS REQUIRES TO INVERT OPERATORS  $L_{S_L}$  AND  $L_{Re}$ .

## Physics-based preconditioner for RMHD (III)

- System is simplified further by:
  - Jacobi iterative scheme in  $\delta\Phi$
  - Schur decomposition
- The preconditioner that results is:

$$\begin{pmatrix} I & \theta D_{Re}^{-1} \vec{B}_0 \cdot \nabla \\ 0 & P_{SI} \end{pmatrix} \begin{pmatrix} \delta\Phi^{m+1} \\ \delta\Psi^{m+1} \end{pmatrix} = \begin{pmatrix} rhs_\Phi^m \\ -G_\Psi + \theta(\vec{B}_0 \cdot \nabla)rhs_\Phi^m \end{pmatrix}$$

where  $P_{SI} = L_{S_L} - \theta^2(\vec{B}_0 \cdot \nabla)D_{Re}^{-1}(\vec{B}_0 \cdot \nabla)$  is the Schur complement of  $\delta\Phi - \delta\Psi$  system.

- Upon termination of the iteration, the vorticity is trivially found from:

$$\delta\omega = \nabla^2 \delta\Phi - G_\Phi.$$

- Only  $P_{SI}$  requires inversion: matrix-light approximate MG techniques:
  - Piece-wise constant restriction.
  - Bilinear prolongation.
  - Matrix-free matrix-vector products.
  - Matrix-light Jacobi smoothing:  $u^{s+1} = u^s + D^{-1}(b - Au^s)$

## Efficiency performance: iteration count

Test problem: resistive (tearing) instability

Grid	$\Delta t(\tau_A)$	$n_{Nt}$	$n_{GM}$	$\frac{\text{GMRES}}{\text{time step}}$	CPU time (s)	$\widehat{CPU}$
$\Delta t = 20\Delta t_{CFL}$						
32x32	1.875	3	2.7	8	12.7	1.6
64x64	0.9375	3	2.5	7.5	120	16
128x128	0.46875	3	2.5	7.5	1128	150
256x256	0.234375	3	3.5	10.4	13563	1304
$\Delta t = 40\Delta t_{CFL}$						
32x32	3.75	3	3.4	10.1	7.5	0.74
64x64	1.875	3	3.3	10	72	7.2
128x128	0.9375	3	3.5	10.6	723	68
256x256	0.46875	3	5	15	9385	625
$\Delta t = 160\Delta t_{CFL}$						
32x32	15	3.5	6.3	22	3.5	0.16
64x64	7.5	3.25	6	19	31	1.6
128x128	3.75	3	6	18	277	15
256x256	1.875	3	10.3	31	4367	141

## Efficiency performance: Explicit vs. implicit

CPU time comparison

Grid	$CPU_{exp}/CPU$	$\Delta t/\Delta t_{exp}$
64x64	4.3	142
128x128	6	294
256x256	7.8	578

NOTE THAT  $\Delta t/\Delta t_{exp} \sim \sqrt{N}$ , AS EXPECTED FROM THE CFL CONDITION

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## 2D incompressible Hall MHD

L. Chacón and D. A. Knoll, *J. Comput. Phys.*, 188 (2), 573-592 (2003)

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## Model

- 2D stream function/vorticity formulation with incompressible ions:

$$\begin{aligned}
 \nabla^2 \Phi &= \omega \\
 (\partial_t + \vec{v} \cdot \nabla - \eta \nabla^2 + \eta_2 \nabla^4) \Psi + E_0 &= d_i \vec{B} \cdot \nabla B_z \\
 (\partial_t + \vec{v} \cdot \nabla - \eta \nabla^2 + \eta_2 \nabla^4) B_z + \dot{S}_{B_z} &= \vec{B} \cdot \nabla v_z - d_i \vec{B} \cdot \nabla (\nabla^2 \Psi) \\
 (\partial_t + \vec{v} \cdot \nabla - \nu \nabla^2) v_z + \dot{S}_{v_z} &= \vec{B} \cdot \nabla B_z \\
 (\partial_t + \vec{v} \cdot \nabla - \nu \nabla^2) \omega + \dot{S}_\omega &= \vec{B} \cdot \nabla (\nabla^2 \Psi)
 \end{aligned}$$

- Supports whistler wave ( $\omega \propto d_i k^2$ ). This wave is dispersive, resulting in very stringent explicit time step limits with mesh refinement.
- Hyperresistivity  $\eta_2$  is chosen so that whistler is damped in shortest scales on the grid. From physical arguments:

$$\eta_2 \propto d_i v_A \Delta^2 ; \quad d_i > \Delta > \pi/k.$$

where  $\Delta$  is the dissipation length. This requires hyperresistivity to be treated implicitly.

## Hall MHD physics-based preconditioner

- Physics-based preconditioner is obtained **simplifying original system by assuming the ordering**:

$$\Delta t_{CFL}(\sim 1/\Delta x^2) \ll \Delta t < \Delta t_{Alfven}(\sim 1/\Delta x)$$

- In this limit, we can **approximate the block Jacobian matrix** as:

$$J_k = \begin{bmatrix} D_\Phi & 0 & 0 & 0 & \textcolor{green}{I} \\ L_{\Phi,\Psi} & D_\Psi & U_{B_z,\Psi} & 0 & 0 \\ L_{\Phi,B_z} & L_{\Psi,B_z} & D_{B_z} & \textcolor{green}{U}_{v_z,B_z} & 0 \\ L_{\Phi,v_z} & L_{\Psi,v_z} & L_{B_z,v_z} & D_{v_z} & 0 \\ L_{\Phi,\omega} & L_{\Psi,\omega} & 0 & 0 & D_\omega \end{bmatrix} \approx \begin{bmatrix} D_\Phi & 0 & 0 & 0 & 0 \\ L_{\Phi,\Psi} & \textcolor{red}{D}_\Psi & \textcolor{red}{U}_{B_z,\Psi} & 0 & 0 \\ L_{\Phi,B_z} & \textcolor{red}{L}_{\Psi,B_z} & \textcolor{red}{D}_{B_z} & 0 & 0 \\ L_{\Phi,v_z} & L_{\Psi,v_z} & L_{B_z,v_z} & L_{B_z,v_z} & D_{v_z} \\ L_{\Phi,\omega} & L_{\Psi,\omega} & 0 & 0 & D_\omega \end{bmatrix}$$

- Red 2x2 block** supports whistler wave, and is **amenable to a Schur decomposition**:

$$\begin{bmatrix} D_\Psi & U_{B_z,\Psi} \\ L_{\Psi,B_z} & D_{B_z} \end{bmatrix} = \begin{bmatrix} I & U_{B_z,\Psi} D_{B_z}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{SC}^\Psi & 0 \\ 0 & D_{B_z} \end{bmatrix} \begin{bmatrix} I & 0 \\ D_{B_z}^{-1} L_{\Psi,B_z} & I \end{bmatrix},$$

where  $P_{SC}^\Psi$  is Schur complement:

$$P_{SC}^\Psi = D_\Psi - U_{\Psi,B_z} D_{B_z}^{-1} L_{\Psi,B_z} \approx \begin{cases} D_\Psi + \Delta t \theta^2 d_i^2 (\vec{B}_{p0} \cdot \nabla) \nabla^2 (\vec{B}_{p0} \cdot \nabla) & (\text{CGSI}) \\ D_\Psi + \Delta t \theta^2 d_i^2 (\vec{B}_{p0} \cdot \nabla)^2 \nabla^2 & (\text{MGSI}) \end{cases}.$$

## CGSI implementation details

- CGSI semi-implicit system:

$$\left[ \frac{1}{\Delta t} + \theta \vec{v}_e \cdot \nabla - \theta \eta \nabla^2 + \theta \eta_2 \nabla^4 + \Delta t \theta^2 d_i^2 (\vec{B} \cdot \nabla) \nabla^2 (\vec{B} \cdot \nabla) \right] \delta \Psi = rhs_{\Psi}$$

- Operator-split solver:

$$\left[ \frac{1}{\Delta t} + \Delta t \theta^2 d_i^2 (\vec{B}_0 \cdot \nabla) \nabla^2 (\vec{B}_0 \cdot \nabla) \right] \delta \Psi^* = rhs_{\Psi} \text{ (CG)}$$

$$\left( \frac{1}{\Delta t} + \theta \vec{v} \cdot \nabla - \theta \eta \nabla^2 \right) \delta \Psi^{**} = \frac{\delta \Psi^*}{\Delta t} \text{ (MG)}$$

$$\left( \frac{1}{\Delta t} + \theta \eta_2 \nabla^4 \right) \delta \Psi^{n+1} = \frac{\delta \Psi^{**}}{\Delta t} \text{ (Jacobi)}$$

- CG is unpreconditioned
- A fixed number of Jacobi iterations ( $\sim 20$ ) is applied in the last step (smoothing)

## MGSI implementation details

- MGSI semi-implicit system:

$$\left[ \frac{1}{\Delta t} + \theta \vec{v}_{e0} \cdot \nabla - \theta \eta \nabla^2 + \theta \eta_2 \nabla^4 + \Delta t \theta^2 d_i^2 (\vec{B}_{p0} \cdot \nabla)^2 \nabla^2 \right] \delta \Psi = rhs_{\Psi}.$$

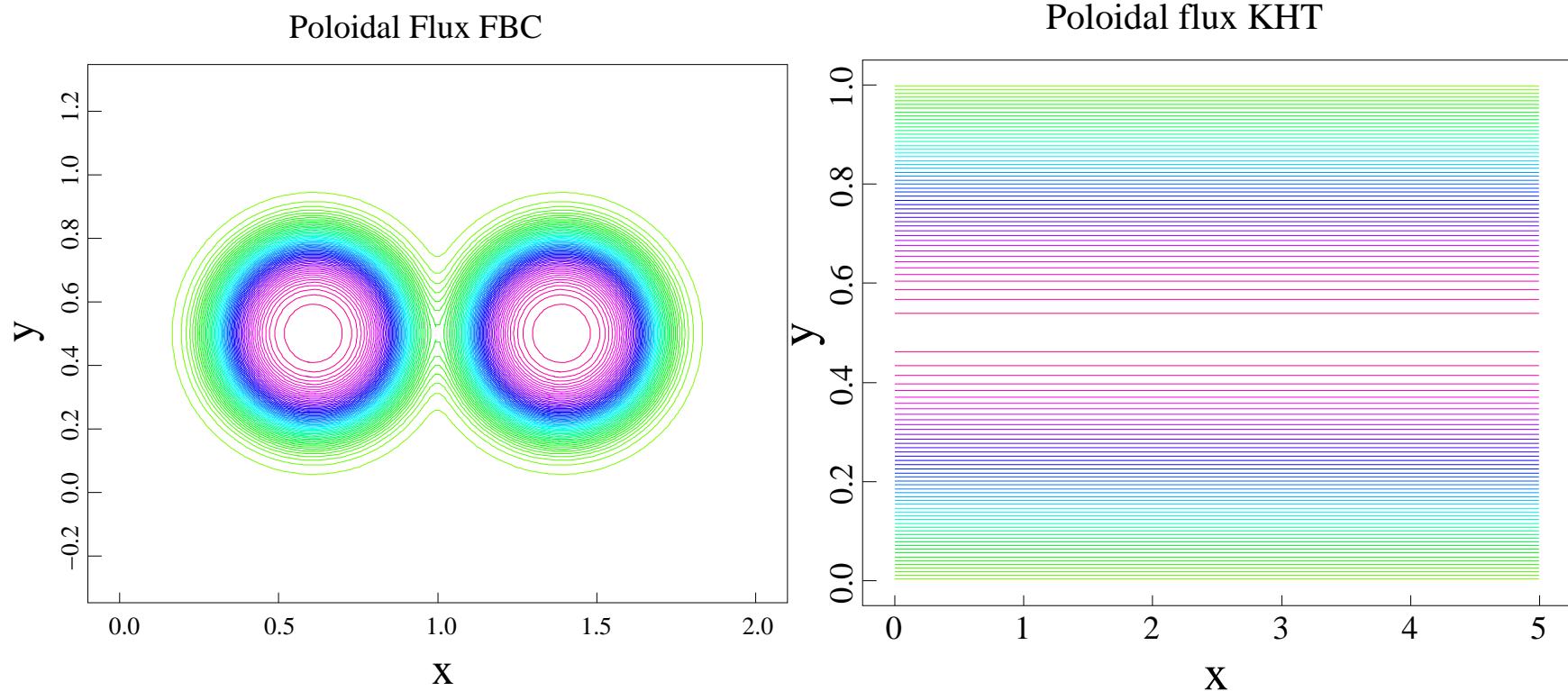
- Coupled MG solver:

$$\begin{aligned} \left[ \frac{1}{\Delta t} + \theta \vec{v}_{e0} \cdot \nabla - \theta \eta \nabla^2 \right] \delta \Psi + \left[ \frac{\eta_2}{\Delta t \theta d_i^2} \nabla^2 + (\vec{B}_{p0} \cdot \nabla)^2 \right] \xi &= rhs_{\Psi}, \\ \Delta t \theta^2 d_i^2 \nabla^2 \delta \Psi - \xi &= 0. \end{aligned}$$

- $\eta_2 / \Delta t d_i^2 \sim \Delta t_{CFL} / \Delta t \sim h \ll 1$
- Matrix-light implementation
- Block Jacobi smoothing

## Test problems

- Flux bundle coalescence (FBC): highly structured magnetic fields
- MHD Kelvin-Helmholtz/tearing instability (KHT): large flows, magnetic field anisotropy



## Efficiency performance: $d_i = 0.2$ , $\Delta t = \Delta t_{Alfven}$

FBC

Grid	$\Delta t$	Newton/ $\Delta t$	GM/ $\Delta t$	CG/GM	CPU (s)	CPU <sub>exp</sub> /CPU	$\Delta t/\Delta t_{CFL}$
CGSI							
64x64	0.02	4.0	1.2	47	12	3.8	74
128x128	0.01	4.0	3.8	106	100	3.9	147
256x256	0.005	4.0	3.8	232	650	5.4	294
MGSI							
64x64	0.02	3.0	0.8	—	14	3.3	74
128x128	0.01	2.6	0.6	—	46	8.5	147
256x256	0.005	2.0	0	—	123	28.0	294

KHT

Grid	$\Delta t$	Newton/ $\Delta t$	GM/ $\Delta t$	CG/GM	CPU (s)	CPU <sub>exp</sub> /CPU	$\Delta t/\Delta t_{CFL}$
CGSI							
64x64	0.02	3	0	4	5	4.2	64
128x128	0.01	3	1.2	13	28	6	125
256x256	0.005	2.8	1.4	24	130	10.5	250
MGSI							
64x64	0.02	3	0.2	—	12	1.8	64
128x128	0.01	3.6	2.4	—	85	2.0	125
256x256	0.005	3.6	2.8	—	367	3.7	250

Efficiency performance:  $d_i = 0.4$ ,  $\Delta t = \Delta t_{Alfven}$

FBC

Grid	$\Delta t$	Newton/ $\Delta t$	GM/ $\Delta t$	CG/GM	CPU (s)	CPU <sub>exp</sub> /CPU	$\Delta t/\Delta t_{CFL}$
CGSI							
64x64	0.02	5.4	16	90	60	1.44	154
128x128	0.01	5.2	15.4	211	416	3.4	294
256x256	0.005	4.8	20	462	3930	1.6	588
MGSI							
64x64	0.02	4.0	4	—	27	3.1	154
128x128	0.01	3.4	2.2	—	80	17.2	294
256x256	0.005	3.0	1.2	—	248	26.0	588

KHT

Grid	$\Delta t$	Newton/ $\Delta t$	GM/ $\Delta t$	CG/GM	CPU (s)	CPU <sub>exp</sub> /CPU	$\Delta t/\Delta t_{CFL}$
CGSI							
64x64	0.02	3.0	0.2	14.8	6	5.8	128
128x128	0.01	3.0	1.0	28.4	32	10.5	250
256x256	0.005	2.8	3.0	70.7	250	10.0	500
MGSI							
64x64	0.02	3.2	1.4	—	16	2.1	128
128x128	0.01	4.0	5.6	—	132	2.5	250
256x256	0.005	4.0	3.8	—	447	5.6	500

Efficiency performance,  $d_i = 0.2$ ,  $\Delta t > \Delta t_{Alfven}$

FBC

$\Delta t/\Delta t_A$	Newton/ $\Delta t$	GM/ $\Delta t$	CPU (s)	CPU <sub>exp</sub> /CPU	$\Delta t/\Delta t_{CFL}$
128x128					
1	2.6	0.6	46	8.5	147
2	3.6	1.8	78	9.4	294
4	4.8	5.8	147	9.3	588
256x256					
1	2	0	123	28.0	294
2	2.8	0.8	214	30.0	588
4	4.2	3.8	460	26.5	1176

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## Road-map to 3D MHD

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## 3D conservative compressible MHD model

Continuity:  $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = D \nabla^2 \rho$

Faraday's law:  $\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0 ; \quad \vec{E} = -\vec{v} \times \vec{B} + \frac{\eta(T)}{\mu_0} \nabla \times \vec{B}$

Momentum:  $\frac{\partial(\rho \vec{v})}{\partial t} + \nabla \cdot \left[ \rho \vec{v} \vec{v} - \frac{\vec{B} \vec{B}}{\mu_0} - \rho \nu(T) \nabla \vec{v} + I(p + \frac{B^2}{2\mu_0}) \right] = 0$

Energy (adiabatic):  $\frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p + \gamma p \nabla \cdot \vec{v} = 0$

Equation of state:  $p = nT$

## Preconditioner is built on approximate formulation

- **Approximations:** non-conservative formulation, density and transport coefficients lagged in time:

$$\text{Faraday's law: } \frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0 \quad ; \quad \vec{E} = -\vec{v} \times \vec{B} + \frac{\eta^n}{\mu_0} \nabla \times \vec{B}$$

$$\text{Momentum: } \rho^n \left[ \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} - \nu^n \nabla \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p$$

$$\text{Energy (adiabatic): } \frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p + \gamma p \nabla \cdot \vec{v} = 0$$

- The preconditioner is based on a linearized, time-discrete formulation:

$$\frac{\delta \vec{B}}{\Delta t} - \theta \nabla \times (\delta \vec{v} \times \vec{B}_0) - \theta \nabla \times (\vec{v}_0 \times \delta \vec{B}) + \theta \nabla \times \left( \frac{\eta^n}{\mu_0} \nabla \times \delta \vec{B} \right) = -\vec{G}_B$$

$$\rho^n \left[ \frac{\delta \vec{v}}{\Delta t} + \theta \vec{v}_0 \cdot \nabla \delta \vec{v} + \theta \delta \vec{v} \cdot \nabla \vec{v}_0 - \theta \nu^n \nabla \delta \vec{v} \right] - \theta \delta \vec{j} \times \vec{B}_0 - \theta \vec{j}_0 \times \delta \vec{B} + \theta \nabla \delta p = -\vec{G}_v$$

$$\frac{\delta p}{\Delta t} + \theta \vec{v}_0 \cdot \nabla \delta p + \theta \delta \vec{v} \cdot \nabla p_0 + \theta \gamma p_0 \nabla \cdot \delta \vec{v} + \theta \gamma \delta p \nabla \cdot \vec{v}_0 = -G_p$$

## Concept of semi-implicit preconditioner

- $\delta\vec{B}$  equation: neglect  $\vec{v}_0$  and resistivity,  $\delta\vec{B} \approx \Delta t[\theta\nabla \times (\delta\vec{v} \times \vec{B}_0) - \vec{G}_B]$
- $\delta p$  equation: neglect  $\vec{v}_0$ ,  $\delta p \approx -\Delta t(\theta\delta\vec{v} \cdot \nabla p_0 + \theta\gamma p_0 \nabla \cdot \delta\vec{v} + G_p)$
- Substitute into  $\delta\vec{v}$  equation, to find a *semi-implicit equation* for  $\delta\vec{v}$ :

$$\left\{ \rho^n \left[ \frac{1}{\Delta t} + \theta\vec{v}_0 \cdot \nabla + \theta(\circ) \cdot \nabla\vec{v}_0 - \theta\nu^n \nabla^2 \right] + \Delta t\theta^2 W(\vec{B}_0, p_0) \right\} \delta\vec{v} = \overrightarrow{rhs}$$

where the SPD operator  $W(\vec{B}_0, p_0)$  is given by:

$$W(\vec{B}_0, p_0)\delta\vec{v} = \vec{B}_0 \times \nabla \times \nabla \times [\delta\vec{v} \times \vec{B}_0] - \vec{j}_0 \times \nabla \times [\delta\vec{v} \times \vec{B}_0] - \nabla[\delta\vec{v} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \delta\vec{v}]$$

and the right hand side  $\overrightarrow{rhs}$  is determined by the substitutions above.

- The semi-implicit operator on  $\delta\vec{v}$  represents a *parabolic system of PDEs* for the three components of  $\delta\vec{v}$ :
  - System is block diagonally dominant.
  - Can be solved simultaneously using a coupled multigrid solver.
- Once a solution for  $\delta\vec{v}$  is found, we find updates for  $\delta\vec{B}$ ,  $\delta p$ , and  $\delta\rho$  including the effects of flow (i.e., with  $\vec{v}_0$  finite).

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## Conclusions

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- Physics-based preconditioning for stiff waves: reformulate hyperbolic couplings into parabolic systems, suitable for MG.
- Connection of SI operator with Schur complement.
- Concept tested for MHD stiff waves: Alfvén ( $\omega \sim k$ ) and whistler ( $\omega \sim k^2$ ).
- The implicit algorithm is scalable ( $\mathcal{O}(N^0)$  number of iterations)
- The implicit algorithm remains accurate for very large time steps ( $\Delta t_{CFL} \ll \Delta t < \tau_{dyn}$ ).  
(NOT SHOWN)
- CPU time gains over explicit methods of about an order of magnitude or more.
- Future work:
  - Explore advanced gridding issues within Newton-Krylov (AMR, adaptive grid).
  - Move toward a 3D primitive-variables implementation of extended MHD.